

NUMERICAL SOLUTIONS OF TIME DEPENDENT PARTIAL DIFFERENTIAL EQUATIONS VIA HAAR WAVELETS

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ABSTRACT. An effective wavelet based scheme coupled with finite difference is used for the solution of two nonlinear time dependent problems namely: Burgers' and Boussinesq equations. These equations have wide-spread application in many fields such as viscous medium, turbulence, fluid dynamics, infiltration phenomena etc. The proposed scheme convert the partial differential equations (PDE) to system of algebraic equations. The obtained system can be solved easily. In this paper convergence of the scheme is also discussed to show validity of the technique. Effectiveness of the scheme is shown with the help of test problems. Numerical results verify that the suggested scheme is more accurate, convenient, fast and require low computational cost.

1. INTRODUCTION

Various important physical phenomenon such as: turbulence problems, shock waves, continuous stochastic etc. are modeled via Burgers' equation given as

$$\partial_t Y + Y \partial_x Y = \frac{1}{R} \partial_{xx} Y, \quad (x, t) \in \Gamma \times \Delta \quad (1.1)$$

with initial conditions

$$Y(x, 0) = \chi(x), \quad x \in \Gamma \quad (1.2)$$

and boundary conditions

$$Y(a, t) = \Upsilon_1(t), \quad Y(b, t) = \Upsilon_2(t) \quad t \in \Delta \quad (1.3)$$

where R is Reynolds number, Γ and Δ are respectively space and time domains. This equation was initiated by Bateman [5] in 1915. Later on in 1948 Burgers [7, 8] studied this equation for several structures of turbulent fluid. Eq. (1.1) is equivalent to Navier-Stokes equations and chiefly containing convection and diffusion terms. The ground water flows serve a vital importance in different fields such as fluid dynamics, biomathematics, chemical engineering and environmental problems. The infiltration phenomena was observed by Boussinesq in 1903. Mathematical form of this process give rise to Boussinesq equation

$$\partial_t Y = Y \partial_{xx} Y + (\partial_x Y)^2, \quad (x, t) \in \Gamma \times \Delta \quad (1.4)$$

coupled with initial condition

$$Y(x, 0) = \chi_1(x), \quad x \in \Gamma \quad (1.5)$$

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and boundary conditions

$$Y(a, t) = \Gamma_1(t), \quad Y(b, t) = \Gamma_2(t) \quad t \in \Delta \quad (1.6)$$

In above two case Burger's equation has been studied by many authors with the help of different numerical methods such as: cubic B-spline finite elements [2], Chebyshev spectral collocation [17], Quartic B-spline differential quadrature [18], finite element [27], a parameter-uniform implicit difference scheme [16], a novel numerical scheme [31], homotopy analysis [30], Semi-Implicit Finite Difference [29], Variational iteration [1] etc.

In last decays wavelets based numerical schemes became popular among the researchers. Initially wavelets based techniques were used for solution of initial and boundary value problems in 1980s. Among different wavelet groups Haar wavelets are the simple one. They are composed of piecewise constant functions which were defined by Alfred Haar in 1910. The attraction of these wavelets is to integrate them in arbitrary times. Haar wavelets are not applicable for the direct solution of differential equations because they are discontinuous at the end point of different intervals. Their derivatives do not exist at the points of discontinuities. To overcome this difficulty Cattani [9] used interpolating splines to regularize these wavelets. The other approach, introduced by Chen and Hsiao [10] in 1997, is integral method. This approach became popular and was used for the solution of different differential equations [3, 9, 12, 11, 24]. Lepik [19, 21] introduced Haar wavelet based collocation method to solve evolution equations and higher order problems. The same author [20] implemented two dimensional Haar wavelets for the solution of PDEs. Jiwari [14] studied Burgers' equation using Haar wavelets coupled with quasilinearization technique. Mittal et al. [23] solved system of viscous Burger's equations using Haar wavelets. Oruc et al. [25, 26] applied Haar wavelets finite difference hybrid method to solve modified Burger's and KdV equations. Recently Somayeh et al. [4] implemented Haar wavelets based scheme for the solution of system of PDEs.

In the present work, we use quasilinearization to linearize nonlinear terms and then θ -weighted scheme coupled with Haar wavelets for the solution of Burgers' and one dimensional Boussinesq equations. The results obtained through proposed numerical scheme are matched with earlier work. Computation shows that present method is reliable, efficient and computationally less expensive.

The paper is arranged in the following way: Preliminaries of Haar wavelets and their integral are discussed in section 2, the scheme is presented in section 3, convergence analysis are shown in section 4 while test problems are reported in section 5.

2. HAAR WAVELETS

To define the Haar wavelets for an arbitrary finite domain $\Gamma = [a, b]$, we refer to the paper [20]. Let us consider $M = 2^\lambda$, where λ denote the maximum resolution level, and $\beta = 2M$. The domain is participated into β subintervals with equal mesh size $\delta x = \frac{b-a}{\beta}$. The 1st and k^{th} Haar wavelets are respectively defined as follows:

$$h_1(x) = \begin{cases} 1 & x \in [a, b] \\ 0 & \text{otherwise.} \end{cases} \quad (2.1)$$

$$h_k(x) = \begin{cases} 1 & x \in [\eta_1(k), \eta_2(k)) \\ -1 & x \in [\eta_2(k), \eta_3(k)) \\ 0, & \text{otherwise} \end{cases} \quad (2.2)$$

where

$$\eta_1(k) = a + 2\zeta\nu\delta x, \quad \eta_2(k) = a + (2\zeta + 1)\nu\delta x, \quad \eta_3(k) = a + 2(\zeta + 1)\nu\delta x, \quad \nu = \frac{M}{\mu},$$

$$\tau = 0, 1, \dots, \lambda, \quad \zeta = 0, 1, \dots, \mu - 1, \quad \mu = 2^T, \quad k = \mu + \zeta + 1.$$

For the solution of n^{th} order differential equation, one need the following iterated integrals

$$P_{n,k}(x) = \int_a^x \int_a^x \cdots \int_a^x h_k(z) dz^n = \frac{1}{(n-1)!} \int_a^x (x-z)^{n-1} h_k(z) dz,$$

$$k = 1, 2, \dots, 2M.$$

Analytical expressions for above integrals are given by

$$P_{n,1}(x) = \frac{(x-a)^n}{n!}. \quad (2.3)$$

$$P_{n,k}(x) = \begin{cases} 0, & x < \eta_1(x) \\ \frac{1}{n!} [x - \eta_1(k)]^n, & x \in [\eta_1(k), \eta_2(k)) \\ \frac{1}{n!} [(x - \eta_1(k))^n - 2((x - \eta_2(k))^n)], & x \in [\eta_2(k), \eta_3(k)) \\ \frac{1}{n!} [(x - \eta_1(k))^n - 2((x - \eta_2(k))^n + (x - \eta_3(k))^n)], & x > \eta_3(k). \end{cases} \quad (2.4)$$

3. METHOD DESCRIPTION

In this section, the proposed method has been discussed which will be used for the solution of nonlinear test examples. In this technique, for the solution of problem (1.1)-(1.3), θ - weighted scheme is used for spacial part whereas finite difference for time derivative as given below

$$\frac{Y^{\xi+1} - Y^\xi}{\Delta t} + \theta (Y \partial_x Y)^{\xi+1} + (1 - \theta) (Y \partial_x Y)^\xi =$$

$$\frac{1}{R} \left[\theta (\partial_{xx} Y)^{\xi+1} + (1 - \theta) (\partial_{xx} Y)^\xi \right] \quad (3.1)$$

the boundary conditions (1.3) takes the form

$$Y(a, t^{\xi+1}) = \Upsilon_1(t^{\xi+1}); \quad Y(b, t^{\xi+1}) = \Upsilon_2^{\xi+1}. \quad (3.2)$$

To linearize the nonlinear term $(Y \partial_x Y)^{\xi+1}$ in Eq. (3.1) applying quasilinearization technique, we obtain

$$(Y \partial_x Y)^{\xi+1} \cong Y^{\xi+1} (\partial_x Y)^\xi + (\partial_x Y)^{\xi+1} Y^\xi - Y^\xi (\partial_x Y)^\xi. \quad (3.3)$$

Using Eq. (3.3) in Eq. (3.1), we get

$$Y^{\xi+1} + \Delta t \left[\theta Y^{\xi+1} (\partial_x Y)^\xi + \theta (\partial_x Y)^{\xi+1} Y^\xi - \frac{\theta}{R} (\partial_{xx} Y)^{\xi+1} \right]$$

$$= Y^\xi + \Delta t (1 - \theta) \left[\frac{1}{R} (\partial_{xx} Y)^\xi - (Y \partial_x Y)^\xi \right] + \Delta t \theta (Y \partial_x Y)^\xi. \quad (3.4)$$

Next, approximating the highest order derivative in Eq. (3.4) by Haar wavelets series

$$(\partial_{xx} Y)^{\xi+1} = \sum_{k=1}^{\beta} \omega_k h_k(x) \quad (3.5)$$

where ω_k are wavelets coefficients to be determine and h_k are Haar wavelets. Integrating Eq. (3.5) twice in domain $[a, x]$ gives

$$\partial_x Y^{\xi+1} = \sum_{k=1}^{\beta} \omega_k P_{1,k}(x) + [\partial_x Y(a)]^{\xi+1} \quad (3.6)$$

$$Y^{\xi+1} = \sum_{k=1}^{\beta} \omega_k P_{2,k}(x) + (x-a) [\partial_x Y(a)]^{\xi+1} + Y^{\xi+1}(a). \quad (3.7)$$

Using boundary condition at $x = b$ (refer to Eq. (3.2)) in Eq. (3.7), the unknown term $[\partial_x Y]^{\xi+1}(a)$ is computed as

$$[\partial_x Y(a)]^{\xi+1} = \Upsilon_2^{\xi+1}(b) - \Upsilon_1^{\xi+1}(a) - \frac{1}{b-a} \sum_{k=1}^{\beta} \omega_k P_{2,k}(b). \quad (3.8)$$

Substituting value from Eq. (3.8) in Eqs. (3.6) and (3.7), one can write

$$[\partial_x Y]^{\xi+1} = \sum_{k=1}^{\beta} \omega_k \left\{ P_{1,k}(x) - \frac{1}{b-a} P_{2,k}(b) \right\} + \Upsilon_2^{\xi+1}(b) - \Upsilon_1^{\xi+1}(a) \quad (3.9)$$

$$Y^{\xi+1} = \sum_{k=1}^{\beta} \omega_k \left\{ P_{2,k}(x) - \frac{x-a}{b-a} P_{2,k}(b) \right\} + (x-a) \left[\Upsilon_2^{\xi+1}(b) - \Upsilon_1^{\xi+1}(a) \right] + \Upsilon^{\xi+1}(a). \quad (3.10)$$

Putting values from Eqs. (3.5), (3.9) and (3.10) in Eq. (3.4) and using the collocation points, $x_m = \frac{m-0.5}{\beta}$, $m = 1, 2, \dots, \beta$, leads to the following system of algebraic equations

$$\begin{aligned} \sum_{k=1}^{\beta} \omega_k [Q(k, m) + \Delta t \theta Q(k, m) (\partial_x Y)_{x=x_m}^{\xi} + \Delta t \theta U(k, m) Y_{x=x_m}^{\xi} \\ - \frac{\theta \Delta t}{R} H(k, m)] = Y_{x=x_m}^{\xi} + \Delta t (1 - \theta) \Gamma + \Delta t \theta (Y \partial_x Y)_{x=x_m}^{\xi} - \Omega (1 + \Delta t \theta) \\ - \Delta t \theta [\Upsilon_2^{\xi+1} - \Upsilon_1^{\xi}] Y_{x=x_m}^{\xi} \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} Q(k, m) &= P_{2,k}(x_m) - \left(\frac{x-a}{b-a} \right) P_{2,k}(b), \quad U(k, m) = P_{1,k}(x) - \frac{1}{b-a} P_{2,k}(b), \\ \Omega &= \left(\frac{x-a}{b-a} \right)_{x=x_m} [\Upsilon_2^{\xi+1}(b) - \Upsilon_1^{\xi+1}(a)] + \Upsilon_1^{\xi+1}(a), \\ \Gamma &= \frac{1}{R} (\partial_{xx} Y)_{x=x_m}^{\xi} - (Y \partial_x Y)_{x=x_m}^{\xi}, \quad H(k, m) = h_k(x_m). \end{aligned}$$

Solving the system (3.11), we get the wavelet coefficients w_k . Once these coefficients are determined, the approximate solution can be obtained from Eq. (3.10). For numerical computation $\theta = \frac{1}{2}$ and the domain $[a, b] = [0, 1]$ have been used.

4. ERROR ANALYSIS

In this section, the error analysis of the current scheme Eq. (3.10) has been examined via asymptotic expansion. Similar procedure can be applied to Eq. (1.4) as well. The following lemma [22] is required for the proof of the convergence theorem.

Lemma 1. (see [22]) If $Y(x) \in L^2(R)$ with $|Y'(x)| \leq \rho$, for all $x \in (0, 1)$, $\rho > 0$ and $Y(x) = \sum_{k=0}^{\infty} \omega_k h_k(x)$. Then $|\omega_k| \leq \frac{\rho}{2^{\tau+1}}$.

Theorem 1. If $Y(x)$ and $Y_\beta(x)$ be exact and approximate solution of Eq. (1.1), then the error norm $\|E_\lambda\|$ at λ^{th} resolution level is

$$\|E_\lambda\| \leq \frac{4\rho}{3} \left(\frac{1}{2^{\lambda+1}} \right)^2. \quad (4.1)$$

Proof: see [13]. □

5. TEST PROBLEMS

In this section, the suggested techniques is applied to selected problems. The obtained results are matched with earlier work which shows effectiveness of the proposed scheme.

Example 1. Case 1: Consider Eq. (1.1) with initial and boundary conditions

$$Y(x, 0) = \sin(\pi x) \quad 0 < x < 1; \quad Y(0, t) = Y(1, t) = 0 \quad t > 0. \quad (5.1)$$

Applying the scheme discussed in section 3 and replacing x by x_m , we get the following system of equations

$$\sum_{k=1}^{\beta} \omega_k \left[Q + \frac{\Delta t}{2} Q (\partial_x Y)^\xi + \frac{\Delta t}{2} U Y^\xi - \frac{\delta t \theta}{R} h_k(x_m) \right] = Y^\xi + \frac{\Delta t}{2R} (\partial_{xx} Y)^\xi \quad (5.2)$$

where

$$Q = P_{2,k}(x_m) - x_m P_{2,k}(1), \quad U = P_{1,k}(x_m) - P_{2,k}(1), \quad m = 1, 2, \dots, \beta.$$

Eq. (5.2) gives system of β equation in β unknowns ω_k .

Case 2: In this case the following initial and boundary conditions are taken with Eq. (1.1)

$$Y(x, 0) = 4x(x - 1), \quad 0 < x < 1, \quad Y(0, t) = Y(1, t) = 0, \quad t > 0. \quad (5.3)$$

The solution has been obtained for different values of R in both cases (Case1 and 2). The results are compared with the results of different methods in literature and are shown in Tables 1-5 (case 1) and 6-9 (case 2). From the tables it is obvious that the present results are better than the other techniques. The solution profiles are plotted in Figures 1-4.

Example 2. In this example, we consider Eq. (1.4) combined with initial and boundary conditions

$$Y(x, 0) = 1 - x^2, \quad 0 < x < 1, \quad Y(0, t) = 1; \quad Y(1, t) = 0, \quad t > 0. \quad (5.4)$$

Applying the proposed scheme, we obtain

$$\begin{aligned} Y^{\xi+1} - \Delta t \theta (Y \partial_{xx} Y)^{\xi+1} - \Delta t \theta \left[(\partial_x Y)^2 \right]^{\xi+1} = \\ Y^\xi + (1 - \theta) \left[(Y \partial_{xx} Y)^\xi + \left((\partial_x Y)^2 \right)^\xi \right]. \end{aligned} \quad (5.5)$$

Linearizing the nonlinear terms by formulas

$$[Y \partial_{xx} Y]^{\xi+1} = Y^{\xi+1} (\partial_{xx} Y)^{\xi} + (\partial_{xx} Y)^{\xi+1} Y^{\xi} - (Y \partial_{xx} Y)^{\xi+1} \quad (5.6)$$

$$\left[(\partial_x Y)^2 \right]^{\xi+1} = 2 (\partial_x Y)^{\xi+1} (\partial_x Y)^{\xi} - \left((\partial_x Y)^2 \right)^{\xi}. \quad (5.7)$$

The approximations of $(\partial_{xx} Y)^{\xi+1}$, $(\partial_x Y)^{\xi+1}$ and, $Y^{\xi+1}$ are given by

$$(\partial_{xx} Y)^{\xi+1} = \sum_{k=1}^{\beta} w_k h_k(x) \quad (5.8)$$

$$(\partial_x Y)^{\xi+1} = \sum_{k=1}^{\beta} w_k [P_{1,k}(x) - P_{2,k}(1)] - 1 \quad (5.9)$$

$$Y^{\xi+1} = \sum_{k=1}^{\beta} w_k [P_{2,k}(x) - x P_{2,k}(1)] - x + 1 \quad (5.10)$$

Using Eqs. (5.6)-(5.10) in Eq. (5.5), we get the following system of equations when $x = x_m$

$$\begin{aligned} & \sum_{k=1}^{\beta} \omega_k \left[Q - \frac{\Delta t}{2} Q (\partial_{xx} Y)^{\xi} - \frac{\Delta t}{2} h_k(x_m) Y^{\xi} - \Delta t U (\partial_x Y)^{\xi} \right] \\ & = Y^{\xi} + \left[x - 1 + \frac{\Delta t}{2} (1 - x) (\partial_{xx} Y)^{\xi} - \Delta t (\partial_x Y)^{\xi} \right], \quad m = 1, 2, \dots, \beta, \end{aligned} \quad (5.11)$$

where Q and U are the same as given in Example 1. Here also Eq. (5.11) generates β number of equation in so many unknowns. The system has been solved for w_k and then approximate solution has been obtained from Eq. (5.10). The results computed, compared with those of finite differences and are listed in Table 10. The results of the present method are in good agreement with those available in literature. Solution profile is plotted in Figure 5.

6. CONCLUDING REMARKS

In this work, Haar wavelets technique is used to compute approximate solutions of Burger's and Boussinesq equations. The obtained solutions are compared with those available in literature. It has been predicted that the results of the present technique are better and more accurate. This method can applied to such type of other problems and can be extended to higher dimensions also.

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TABLE 1. Comparison of present method, exact and method available in literature of example 1 (case 1) when $T = 0.01$, $\Delta t = 0.0001$, $R = 1$

x	[15]	[28]	Present ($\lambda = 5$)	Present ($\lambda = 7$)	Exact
0.1	0.11460	0.11454	0.11460	0.11461	0.11461
0.2	0.21813	0.21802	0.21814	0.21816	0.21816
0.3	0.30057	0.30041	0.30058	0.30061	0.30062
0.4	0.35385	0.35366	0.35385	0.35389	0.35390
0.5	0.37264	0.37245	0.37265	0.37269	0.37270
0.6	0.35497	0.35478	0.35497	0.35501	0.35502
0.7	0.30238	0.30222	0.30239	0.30242	0.30243
0.8	0.21994	0.21983	0.21995	0.21997	0.21997
0.9	0.11572	0.11565	0.11571	0.11573	0.11573

TABLE 2. Comparison of present method, exact and method available in literature of example 1 (case 1) when $T = 0.1$, $\Delta t = 0.001$, $R = 1$

x	[15]	[28]	Present ($\lambda = 5$)	Present ($\lambda = 7$)	Exact
0.1	0.10952	0.10947	0.10953	0.10953	0.10954
0.2	0.20976	0.20965	0.20977	0.20978	0.20979
0.3	0.29186	0.2917	0.29187	0.29189	0.29190
0.4	0.34788	0.34769	0.34789	0.34791	0.34792
0.5	0.37153	0.37133	0.37153	0.37157	0.37158
0.6	0.35900	0.35881	0.35900	0.35904	0.35905
0.7	0.30986	0.3097	0.30986	0.30990	0.30991
0.8	0.22779	0.22766	0.22778	0.22781	0.22782
0.9	0.12067	0.12061	0.12067	0.12068	0.12069

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TABLE 3. Comparison of present method, exact and method available in literature of example 1 (case 1) when $\Delta t = 0.01$, $R = 10$

x	T	[15]	[28]	Present ($\lambda = 5$)	Present ($\lambda = 7$)	Exact
0.25	0.4	0.30881	0.30866	0.30889	0.30888	0.30889
	0.6	0.24069	0.24058	0.24075	0.24073	0.24074
	1.0	0.16254	0.16248	0.16258	0.16256	0.16256
	3.0	0.02720	0.02719	0.02720	0.02720	0.02720
0.5	0.4	0.56955	0.56934	0.56967	0.56962	0.56963
	0.6	0.44714	0.44697	0.44725	0.44720	0.44721
	1.0	0.29188	0.29177	0.29195	0.29191	0.29192
	3.0	0.04021	0.04018	0.04020	0.040204	0.04020
0.75	0.4	0.62540	0.62523	0.62547	0.62546	0.62544
	0.6	0.48715	0.48700	0.48724	0.48722	0.48721
	1.0	0.28744	0.28732	0.28747	0.28747	0.28747
	3.0	0.02978	0.02976	0.02976	0.02977	0.02877

TABLE 4. Comparison of present method, exact and method available in literature of example 1 (case 1) when $\Delta t = 0.01$, $R = 100$

x	T	[15]	[28]	Present ($\lambda = 5$)	Present ($\lambda = 7$)	Exact
0.25	0.4	0.34229	0.34267	0.34192	0.34191	0.34191
	0.6	0.26902	0.26908	0.26896	0.26896	0.26896
	1.0	0.18817	0.18806	0.18819	0.18819	0.18819
	3.0	0.07511	0.07505	0.07511	0.07511	0.07511
0.5	0.4	0.66797	0.67588	0.66074	0.66072	0.66071
	0.6	0.53211	0.53678	0.52943	0.52942	0.52942
	1.0	0.37500	0.37671	0.37442	0.37442	0.37442
	3.0	0.15018	0.15022	0.15017	0.15017	0.15018
0.75	0.4	0.93680	0.95424	0.91041	0.91034	0.91026
	0.6	0.77724	0.79252	0.76731	0.76727	0.76724
	1.0	0.55833	0.56535	0.55607	0.55605	0.55605
	3.0	0.22485	0.22528	0.22484	0.22481	0.22481

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TABLE 5. Comparison of present method, exact and method available in literature of example 1 (case 1) when $\Delta t = 0.01$, $R = 200$

x	T	[15]	[28]	Present ($\lambda = 5$)	Present ($\lambda = 7$)	Exact
0.25	1	0.19625	0.18941	0.18879	0.18878	0.18879
	5	0.04741	0.04692	0.04696	0.04696	0.04741
	10	0.02434	0.0242	0.02421	0.02421	0.02434
	15	0.01636	0.0163	0.01630	0.01630	0.01636
0.5	1	0.39072	0.3873	0.37573	0.37572	0.38802
	5	0.09482	0.094	0.09392	0.09392	0.09481
	10	0.04866	0.04841	0.04842	0.04842	0.04866
	15	0.03255	0.03243	0.03244	0.03243	0.03255
0.75	1	0.58149	0.60018	0.55840	0.55839	0.57256
	5	0.14218	0.14135	0.14083	0.14083	0.14215
	10	0.07151	0.07117	0.07117	0.07113	0.07152
	15	0.04432	0.04413	0.04415	0.04413	0.04433

TABLE 6. Comparison between present results, exact and those available in literature for example 1 (case 2) when $T = 0.1$, $\Delta t = 0.001$, $R = 1.0$

x	[15]	[28]	Present ($\lambda = 5$)	Present ($\lambda = 7$)	Exact
0.1	0.11288	0.11286	0.11288	0.11289	0.11289
0.2	0.21622	0.21619	0.21623	0.21624	0.21625
0.3	0.30092	0.30088	0.30094	0.30096	0.30097
0.4	0.35881	0.35877	0.35882	0.35885	0.35886
0.5	0.38337	0.38332	0.38338	0.38341	0.38342
0.6	0.37061	0.37056	0.37061	0.37065	0.37066
0.7	0.32002	0.31998	0.32002	0.32006	0.32007
0.8	0.23534	0.23531	0.23533	0.23536	0.23537
0.9	0.12470	0.12469	0.12470	0.12471	0.12472

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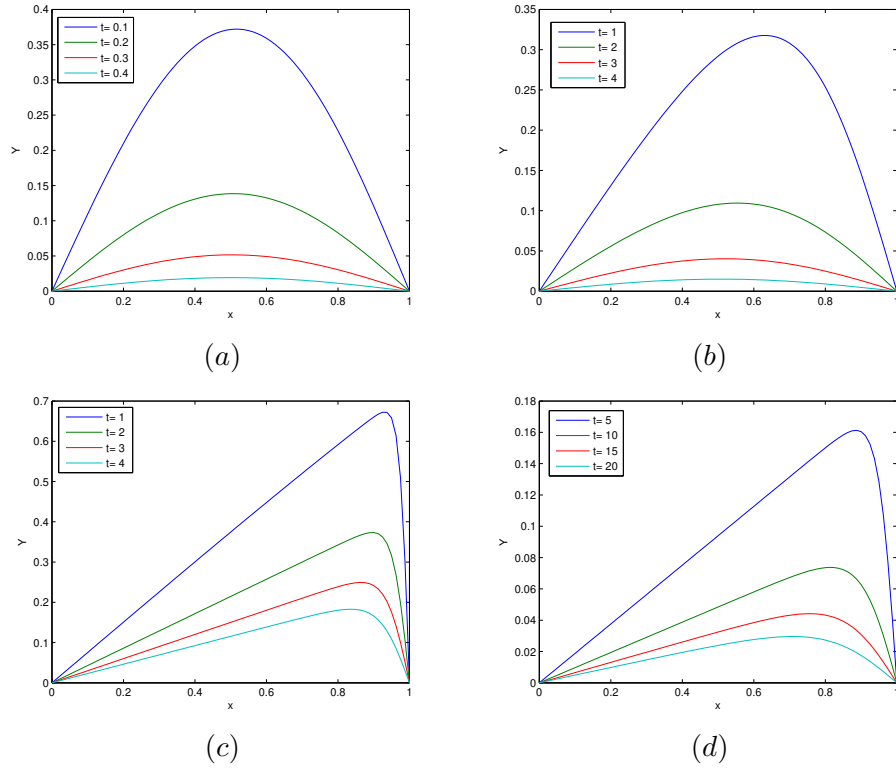


FIGURE 1. Solution profiles of example 1 (case 1) when (a) : $R = 1, \Delta t = 0.001$, $R = 10, \Delta t = 0.01$, $R = 100, \Delta t = 0.01$, $R = 200, \Delta t = 0.01$, $\delta x = 0.0125$

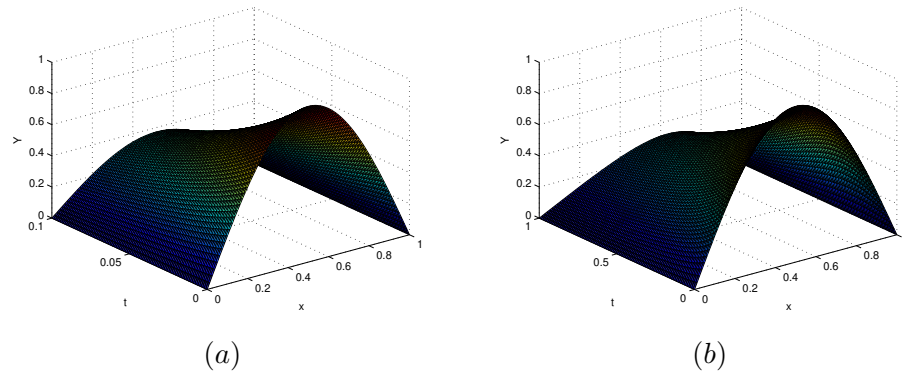


FIGURE 2. 3D plot of example 1 (case 1) when (a) : $Re = 1, \Delta t = 0.001$, (b) : $Re = 10, \Delta t = 0.01$

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TABLE 7. Comparison between present results, exact and those available in literature for example 1 (case 2) when $\Delta t = 0.01$, $R = 10$

x	T	[15]	[28]	Present ($\lambda = 5$)	Present ($\lambda = 7$)	Exact
0.25	0.4	0.31743	0.31735	0.31752	0.31751	0.31752
	0.6	0.24609	0.24603	0.24614	0.24613	0.24614
	1.0	0.16558	0.16554	0.16561	0.16559	0.16560
	3.0	0.02776	0.02775	0.02775	0.02775	0.02776
0.5	0.4	0.58446	0.58441	0.58457	0.58453	0.58454
	0.6	0.45791	0.45786	0.45802	0.45797	0.45798
	1.0	0.29831	0.29826	0.29838	0.29834	0.29834
	3.0	0.04107	0.04105	0.04106	0.04106	0.04106
0.75	0.4	0.64558	0.6457	0.64566	0.64563	0.64562
	0.6	0.50261	0.50265	0.50270	0.50268	0.50268
	1.0	0.29582	0.2958	0.29585	0.29585	0.29586
	3.0	0.03044	0.03043	0.03043	0.03043	0.03044

TABLE 8. Comparison between present results, exact and those available in literature for example 1 (case 2) when $\Delta t = 0.01$, $R = 100$

x	T	[15]	[28]	Present ($\lambda = 5$)	Present ($\lambda = 7$)	Exact
0.25	0.4	0.36273	0.36339	0.36226	0.36226	0.36226
	0.6	0.28212	0.28228	0.28203	0.28203	0.28204
	1.0	0.19467	0.19458	0.19469	0.19469	0.19469
	3.0	0.07613	0.07607	0.07613	0.07613	0.07613
0.5	0.4	0.69186	0.70088	0.68371	0.68369	0.68368
	0.6	0.55125	0.55671	0.54833	0.54832	0.54832
	1.0	0.38627	0.38826	0.38568	0.38567	0.38568
	3.0	0.15218	0.15223	0.15218	0.15218	0.15218
0.75	0.4	0.94940	0.96667	0.92061	0.92056	0.92050
	0.6	0.79399	0.81017	0.78305	0.78302	0.78299
	1.0	0.57170	0.57942	0.56933	0.56932	0.56932
	3.0	0.22778	0.22824	0.22777	0.22774	0.22774

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TABLE 9. Comparison between present results, exact and those available in literature for example 1 (case 2) when $\Delta t = 0.01$, $R = 200$

x	T	[15]	Present ($\lambda = 5$)	Present ($\lambda = 7$)	Exact
0.25	1	0.19625	0.19608	0.19608	0.18879
	5	0.04741	0.04741	0.04741	0.04741
	10	0.02434	0.02433	0.02433	0.02434
	15	0.01636	0.01636	0.01636	0.01636
0.5	1	0.39072	0.38802	0.38801	0.38802
	5	0.09482	0.09481	0.09481	0.09481
	10	0.04866	0.04866	0.04865	0.04866
	15	0.03255	0.03255	0.03255	0.03255
0.75	1	0.58149	0.57257	0.57257	0.57256
	5	0.14218	0.14216	0.14215	0.14215
	10	0.07151	0.07155	0.07151	0.07152
	15	0.04432	0.04435	0.04432	0.04433

TABLE 10. Comparison of present results with those available in literature of example 2 when $T = 0.0125$, $\Delta t = 0.00025$, $R = 1.0$

x	[6]	Present ($\lambda = 5$)	Present ($\lambda = 7$)
0.0	1.000000	1.000000	1.000000
0.1	0.974726	0.976778	0.974729
0.2	0.940947	0.944126	0.940951
0.3	0.892820	0.896005	0.892822
0.4	0.827388	0.829832	0.827389
0.5	0.743626	0.744915	0.743626
0.6	0.641297	0.641141	0.641297
0.7	0.520367	0.518500	0.520367
0.8	0.380833	0.376990	0.380832
0.9	0.222816	0.216656	0.222691
1.0	0.000000	0.000000	0.000000

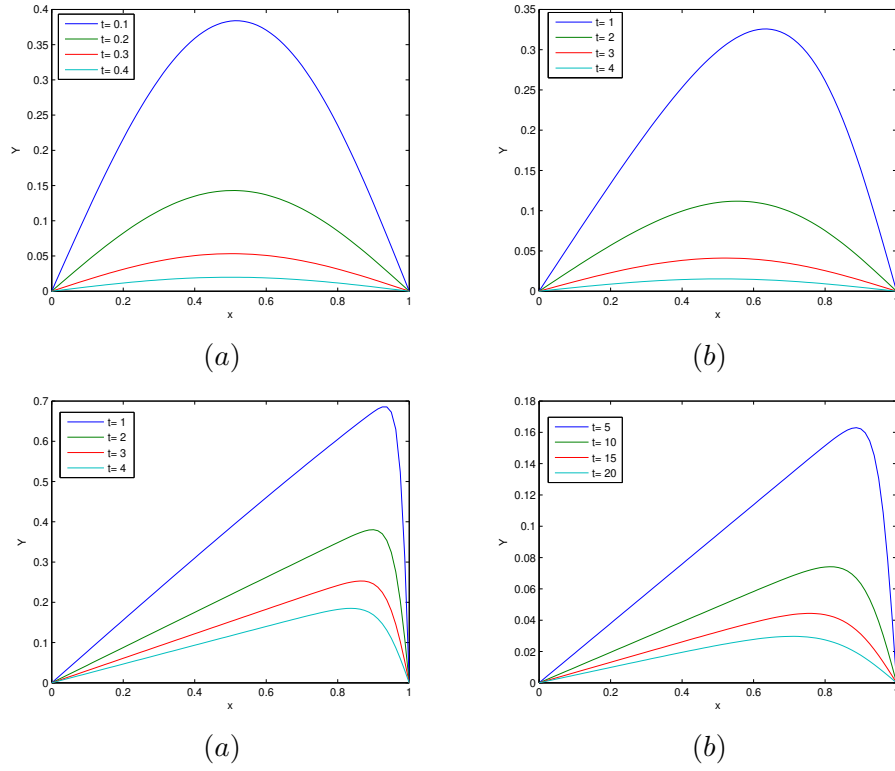


FIGURE 3. Solution profile of example 1 (case 2) when (a) : $Re = 1, \Delta t = 0.001, Re = 10, \Delta t = 0.01, Re = 100, \Delta t = 0.01, Re = 200, \Delta t = 0.01, \delta x = 0.0125$

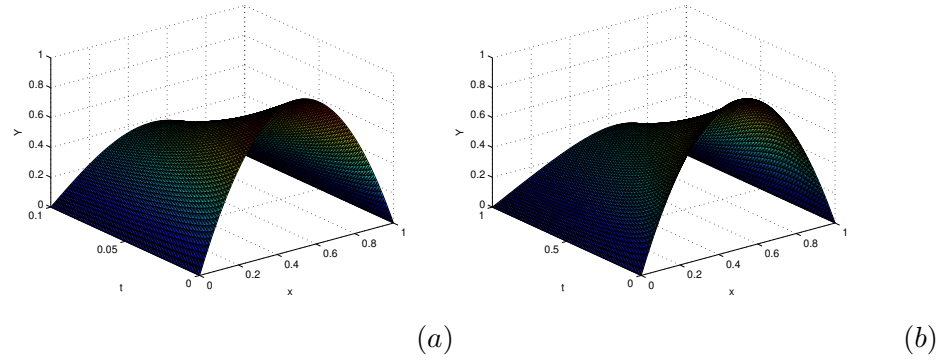


FIGURE 4. 3D plot of example 1 (case 2) when (a) : $Re = 1, \Delta t = 0.001$, (b) : $Re = 10, \Delta t = 0.01$

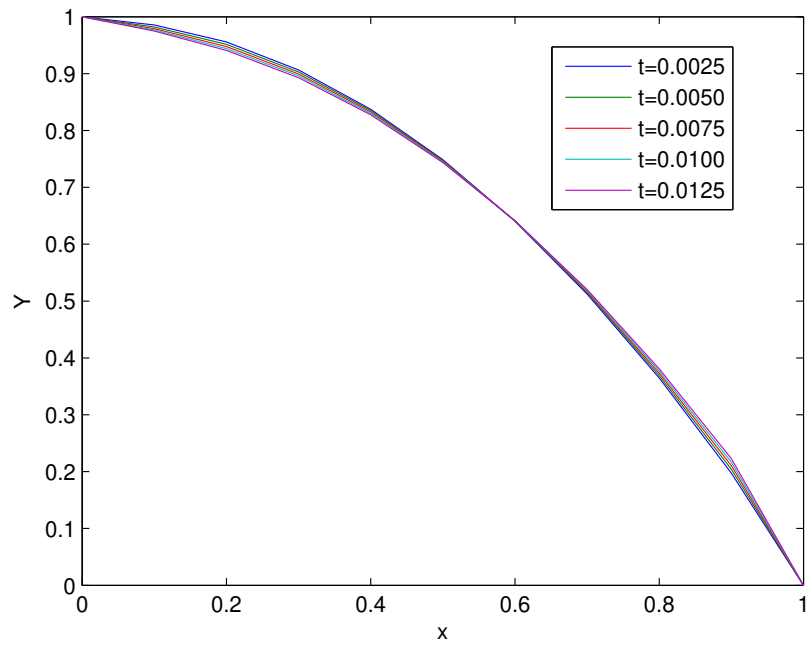


FIGURE 5. Solution profile of example 2 for different times.